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A Sharp Estimate of the Landau Constants

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1. INTRODUCTION

Let $A(\mathscr{D})$ be the Banach space of functions continuous in the closed unit disc $\mathscr{D} = \{z: |z| \leqslant 1\}$ and analytic in the interior of $\mathscr{D}$, equipped with the sup norm. Denote by π_n the subspace of $A(\mathscr{D})$ consisting of all polynomials of degree $\leqslant n$. An operator $P_n: A(\mathscr{D}) \to \pi_n$ is called a projection if it is a linear bounded idempotent operator. The family of all projections from $A(\mathscr{D})$ onto π_n will be denoted by Σ. A projection P_n^* is called an optimal (or minimal) projection if $\|P_n^*\| \leqslant \|P_n\|$ for any $P_n \in \Sigma$.

It has been proven by Geddes and Mason [4] that the Taylor projection $T_n: A(\mathscr{D}) \to \pi_n$, defined as an n-partial sum of Taylor series, is of minimum norm. Recently [2] Fisher *et al.* have shown the Taylor projection to be the unique polynomial projection of minimum norm. An exact expression for this norm $\|T_n\|$ has been established by Landau [6]:

$$\|T_n\| \equiv G_n = 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 + \cdots + \left(\frac{1 \cdot 3 \cdots 2n-1}{2 \cdot 4 \cdots 2n}\right)^2. \quad (1)$$

Landau has also shown that

$$G_n \sim \frac{1}{\pi} \log n.$$

An investigation of the asymptotic behaviour of G_n has been continued by Watson [8]. He has proved in particular that

$$G_n = \frac{1}{\pi} \log(n+1) + A - \delta_n, \quad (2)$$

where

$$A = \frac{1}{\pi}(\gamma + 4 \log 2) = 1.0662, \quad (3)$$

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γ is Euler's constant and

$$\lim_{n\to\infty} \delta_n = 0.$$

Watson gives also the following estimate:

$$G_n < \frac{2}{\pi}\int_0^{\pi/2} \frac{|\sin(n+1)\theta|}{\sin\theta}\, d\theta \equiv \rho_n, \tag{4}$$

ρ_n being the nth classical Lebesgue constant. In view of Fejér's result [1]

$$\rho_n = \frac{4}{\pi^2}\log(n+1) + C + o(1),$$

estimate (4) seems to be conservative.

The purpose of this note is to establish more precise estimates for G_n.

2. Results

THEOREM. *Let* $\delta_n = (1/\pi)\log(n+1) + A - G_n$, *where* A *is the constant defined by* (3). *Then the sequence* $\{\delta_n\}$ *is monotone decreasing. Moreover, for* $n = 0, 1, 2,...,$

$$1 + \frac{1}{\pi}\log(n+1) \leqslant G_n < 1.0663 + \frac{1}{\pi}\log(n+1). \tag{5}$$

Proof. To prove that $\{\delta_n\}$ is monotone decreasing we need to verify that

$$G_n - G_{n-1} \geqslant \frac{1}{\pi}\log\frac{n+1}{n}, \qquad n = 1, 2,...,$$

or, in view of the representation (1), that

$$\left[\frac{(2n-1)!!}{(2n)!!}\right]^2 \geqslant \frac{1}{\pi}\log\frac{n+1}{n}, \tag{6}$$

where

$$(2n)!! = 2\cdot 4\cdot \cdots \cdot (2n-2)\cdot(2n),$$
$$(2n-1)!! = 1\cdot 3\cdot 5\cdot \cdots \cdot (2n-3)(2n-1).$$

To prove (6) we appeal to the following stronger[1] Wallis' formula

[1] Note that the classical Wallis' formula is not fine enough to prove estimate (6).

established by Gurland in [5] as a consequence of a basic theorem of mathematical statistics:

$$\frac{4n+3}{(2n+1)^2}\left\{\frac{(2n)!!}{(2n-1)!!}\right\}^2 < \pi < \frac{4}{4n+1}\left\{\frac{(2n)!!}{(2n-1)!!}\right\}^2. \qquad (7)$$

Thus to establish (6) it is sufficient to verify that

$$\frac{4n+3}{(2n+1)^2} \geqslant \log\frac{n+1}{n}, \qquad n = 1, 2, 3, \ldots . \qquad (8)$$

We put $1/n = x$ and show that for any $x \in [0, 1]$,

$$\frac{x(3x+4)}{(x+2)^2} \geqslant \log(1+x). \qquad (8')$$

Let now

$$R(x) = \frac{3x^2+4x}{(x+2)^2} - \log(1+x).$$

An easy computation reveals

$$R'(x) = \frac{-x(x^2-2x-4)}{(x+2)^3(1+x)} \geqslant 0, \qquad x \in [0, 1],$$

and since $R(0) = 0$, (8′) follows.

Finally, $\delta_0 = A - 1$ which completes the proof of the theorem.

Remark. The above estimate (5) may be compared with two other estimates which are of special importance in approximation theory. The first, due to Galkin [3], for the classical Lebesgue constants:

$$0.9894 + \frac{4}{\pi^2}\log(2n+1) < \rho_n \leqslant 1 + \frac{4}{\pi^2}\log(2n+1)$$

and the second, due to Rivlin [7], concerning the norm of the interpolation operator at the Chebyshev nodes:

$$0.9625 + \frac{2}{\pi}\log(n+1) < \Lambda_n(T) \leqslant 1 + \frac{2}{\pi}\log(n+1).$$

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